

WIENER'S LEMMA FOR INFINITE MATRICES II

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ABSTRACT. In this paper, we introduce a class of infinite matrices related to the Beurling algebra of periodic functions, and we show that it is an inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$, the algebra of all bounded linear operators on the weight sequence space ℓ_w^q , for any $1 \leq q < \infty$ and any discrete Muckenhoupt A_q -weight w .

1. INTRODUCTION

Let us begin the sequel to [43] by introducing a new class of infinite matrices,

$$(1.1) \quad \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},$$

where $d \geq 1$, $|x|_\infty := \max(|x_1|, \dots, |x_d|)$ for $x := (x_1, \dots, x_d) \in \mathbb{R}^d$, and

$$(1.2) \quad \|A\|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)} := \sum_{k \in \mathbb{Z}^d} \left(\sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \right).$$

It is observed that a Toeplitz matrix $A := (a(i - j))_{i, j \in \mathbb{Z}}$ associated with a sequence $a := (a(n))_{n \in \mathbb{Z}}$ belongs to $\mathcal{B}(\mathbb{Z}, \mathbb{Z})$ if and only if the Fourier series $\hat{a}(\xi) := \sum_{n \in \mathbb{Z}} a(n) \exp(-\sqrt{-1} n\xi)$ belongs to the algebra

$$(1.3) \quad A^*(\mathbb{T}) := \left\{ \sum_{n=-\infty}^{\infty} a(n) e^{-\sqrt{-1} n\xi} \mid \sum_{k=0}^{\infty} \sup_{|n| \geq k} |a(n)| < \infty \right\}.$$

The above algebra $A^*(\mathbb{T})$ was introduced by A. Beurling for establishing contraction properties of periodic functions [8], and was used in considering pointwise summability of Fourier series [10, 15, 16, 40, 48]. So the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices can be interpreted as a non-commutative matrix extension of the *Beurling algebra* $A^*(\mathbb{T})$.

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Define the *Gröchenig-Schur class* $\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$(1.4) \quad \mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \left(\sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |a(i, j)|, \sup_{j \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |a(i, j)| \right) < \infty \right\}$$

[25, 37, 41, 43], and the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ by

$$(1.5) \quad \mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \sum_{k \in \mathbb{Z}^d} \left(\sup_{i-j=k} |a(i, j)| \right) < \infty \right\}$$

[6, 20, 25, 32, 39, 43]. The above two classes of infinite matrices appeared in the study of Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 24, 28, 39, 42] for a sample of papers). From (1.1), (1.4) and (1.5) it follows that

$$(1.6) \quad \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d).$$

This shows that any matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ belongs to the Gröchenig-Schur class $\mathcal{S}(\mathbb{Z}^d, \mathbb{Z}^d)$ and also the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$.

An equivalent way of defining $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is the existence of a radially decreasing sequence $\{b(i)\}_{i \in \mathbb{Z}^d}$ for any infinite matrix $A := (a(i, j))_{i, j \in \mathbb{Z}^d} \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ such that

$$(1.7) \quad |a(i, j)| \leq b(i - j) \quad \text{for all } i, j \in \mathbb{Z}^d,$$

$$(1.8) \quad \sum_{i \in \mathbb{Z}^d} b(i) < \infty,$$

and

$$(1.9) \quad b(i) = h(|i|_\infty) \text{ for some decreasing sequence } \{h(n)\}_{n=0}^\infty.$$

Therefore any infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable, radial and (radially) decreasing sequence. We remark that any infinite matrix in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable sequence [6, 20, 25, 32, 39, 43].

A positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ is said to be a *discrete A_q -weight* for $1 < q < \infty$ if there exists a positive constant C such that

$$(1.10) \quad \left(N^{-d} \sum_{i \in a + [0, N-1]^d} w(i) \right) \left(N^{-d} \sum_{i \in a + [0, N-1]^d} (w(i))^{-(q-1)} \right)^{1/(q-1)} \leq C$$

hold for all $a \in \mathbb{Z}^d$ and $1 \leq N \in \mathbb{Z}$, and to be a *discrete A_1 -weight* if there exists a positive constant C such that

$$(1.11) \quad N^{-d} \sum_{i \in a + [0, N-1]^d} w(i) \leq C \inf_{i \in a + [0, N-1]^d} w(i)$$

hold for all $a \in \mathbb{Z}^d$ and $1 \leq N \in \mathbb{Z}$ [18, 40]. The smallest constant C for which (1.10) holds when $1 < q < \infty$, and respectively for which (1.11) holds when $q = 1$, to be denoted by $A_q(w)$, is the *discrete A_q -bound*. The positive sequences $((1 + |i|_\infty)^\alpha)_{i \in \mathbb{Z}^d}$ with $-d < \alpha < d(q-1)$ if $1 < q < \infty$, and respectively with $-d < \alpha \leq 0$ if $q = 1$, are discrete A_q -weights.

For $1 < q < \infty$, a positive locally integrable function w on \mathbb{R}^d is said to be an *A_q -weight* if there exists a positive constant C such that

$$(1.12) \quad \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-(q-1)} dx \right)^{1/(q-1)} \leq C$$

for all cubic $Q \subset \mathbb{R}^d$, where $|Q|$ denotes the Lebesgue measure of the cubic Q [36]. Similarly for $q = 1$, a positive locally integrable function w is said to be an *A_1 -weight* if there exists a positive constant C such that

$$(1.13) \quad \frac{1}{|Q|} \int_Q w(y) dy \leq C w(x), \quad x \in Q$$

for all cubic $Q \subset \mathbb{R}^d$ [14]. One may then verify that a positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete A_q -weight if and only if $\tilde{w}(x) := \sum_{i \in \mathbb{Z}^d} w(i) \chi_{[-1/2, 1/2]^d}(x - i)$ is an A_q -weight, where $1 \leq q < \infty$ and χ_E is the characteristic function on a set E [33].

For $1 \leq q < \infty$ and a positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ on \mathbb{Z}^d , let $\ell_w^q := \ell_w^q(\mathbb{Z}^d)$ be the space of all weighted q -summable sequences on \mathbb{Z}^d , i.e.,

$$(1.14) \quad \ell_w^q(\mathbb{Z}^d) := \left\{ (c(i))_{i \in \mathbb{Z}^d} \mid \|c\|_{q,w} := \left(\sum_{i \in \mathbb{Z}^d} |c(i)|^q w(i) \right)^{1/q} < \infty \right\}.$$

For the trivial weight w_0 (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$), we will use ℓ^q and $\|\cdot\|_q$ instead of $\ell_{w_0}^q$ and $\|\cdot\|_{q,w_0}$ for brevity. Define the *discrete maximal function* by

$$(1.15) \quad Mc(i) := \sup_{0 \leq N \in \mathbb{Z}} \frac{1}{(2N+1)^d} \sum_{k \in i + [-N, N]^d} |c(k)| \quad \text{for } c := (c(i))_{i \in \mathbb{Z}^d}.$$

Similar to the characterization of an A_q -weight on \mathbb{R}^d via the standard maximal operator [34], the discrete A_q -weight can be characterized via

the discrete maximal function on the weighted ℓ^q space. More precisely, a positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete A_q -weight if and only if the discrete maximal operator $c \mapsto Mc$ is of weak-type (ℓ_w^q, ℓ_w^q) , i.e., there exists a positive constant C such that

$$(1.16) \quad \sum_{Mc(i) \geq \alpha} w(i) \leq \frac{C}{\alpha^q} \|c\|_{q,w}^q \quad \text{for all } \alpha > 0 \text{ and } c \in \ell_w^p.$$

Moreover for $1 < q < \infty$, the discrete maximal operator M is of strong type (ℓ_w^q, ℓ_w^q) for a discrete A_q -weight w , i.e., there exists a positive constant C' such that

$$(1.17) \quad \|Mc\|_{q,w} \leq C' \|c\|_{q,w} \quad \text{for all } c \in \ell_w^p.$$

The reader may refer to [18] for a complete account of the theory of weighted inequalities and its ramification.

Now let us present our results for infinite matrices in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. In Section 3, we establish the following algebraic properties for the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices.

Theorem 1.1. *Let $1 \leq q < \infty$, and let w be a discrete A_q -weight. Then $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is a unital Banach algebra under matrix multiplication, and also a subalgebra of $\mathcal{B}(\ell_w^q)$, the algebra of all bounded linear operators on the weight sequence space ℓ_w^q .*

By Theorem 1.1, every infinite matrix $A \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ defines a bounded operator on ℓ_w^q for any $1 \leq q < \infty$ and for any discrete A_q -weight w , i.e., there exists a positive constant C such that

$$(1.18) \quad \|Ac\|_{q,w} \leq C \|c\|_{q,w} \quad \text{for all } c \in \ell_w^q.$$

Besides the above boundedness of an infinite matrix on the weighted sequence space ℓ_w^q , it is natural to consider ℓ_w^q -stability. Here for $1 \leq q < \infty$ and a positive sequence w on \mathbb{Z}^d , we say that an infinite matrix A has ℓ_w^q -stability if there exists a positive constant C such that

$$(1.19) \quad C^{-1} \|c\|_{q,w} \leq \|Ac\|_{q,w} \leq C \|c\|_{q,w} \quad \text{for all } c \in \ell_w^q.$$

The ℓ_w^q -stability is one of the basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 24, 28, 29, 39, 42, 43] and the references therein.) In Section 4, we establish the equivalence of ℓ_w^q -stabilities of any infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ for different exponents $1 \leq q < \infty$ and for different discrete A_q -weights w .

Theorem 1.2. *Let $A \in \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$. If A has ℓ_w^q -stability for some $1 \leq q < \infty$ and for some discrete A_q -weight w , then it has $\ell_w^{q'}$ -stability for all $1 \leq q' < \infty$ and for all discrete $A_{q'}$ -weights w' .*

The reader may refer to [1, 38, 49] for the equivalence of unweighted ℓ_w^q -stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. If $A \in \mathcal{B}(\ell_w^q)$ has a *left inverse* $B \in \mathcal{B}(\ell_w^q)$, i.e., $BA = I$, then A has ℓ_w^q -stability. The converse is not true in general, unless $q = 2$. As an application of Theorem 1.2, we show that the converse holds for any infinite matrix A in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.

Corollary 1.3. *Let $1 \leq q < \infty$, and let w be a discrete A_q -weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ has ℓ_w^q -stability if and only if it has a left inverse in $\mathcal{B}(\ell_w^q)$.*

Given a Banach algebra \mathcal{B} , a subalgebra \mathcal{A} of \mathcal{B} is said to be *inverse-closed* if $A \in \mathcal{A}$ and the inverse A^{-1} of the element A exists in \mathcal{B} implies that $A^{-1} \in \mathcal{A}$ [19, 35, 47]. The next question following the ℓ_w^q -stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is whether its inverse, if exists in $\mathcal{B}(\ell_w^q)$, belongs to $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, or in the other word, whether $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$.

The inverse-closedness for the subalgebra of absolutely convergent Fourier series in the algebra of bounded periodic functions was first studied in [9, 19, 35, 50]. The inverse-closed property (=Wiener's lemma) has been established for infinite matrices satisfying various off-diagonal decay conditions, see [3, 5, 6, 17, 23, 25, 29, 39, 41, 43] for a sample of papers. Inverse-closedness occurs under various names (such as spectral invariance, Wiener pair, local subalgebra) in many fields of mathematics, see the survey [21].

The inverse-closed property for non-commutative matrix subalgebra has been shown to be crucial for the well-localization of dual wavelet frames and dual Gabor frames [4, 24, 29, 30, 31], the algebra of pseudo-differential operators [22, 28, 39], the fast implementation in numerical analysis [12, 13, 27], and the local reconstruction in sampling theory [2, 42, 45].

It mixes art and hard mathematical work to consider the inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$. The art is to guess the off-diagonal decay of infinite matrices in an algebra \mathcal{A} , while the work is to prove the inverse-closedness of the algebra \mathcal{A} in $\mathcal{B}(\ell_w^q)$. There are several approaches to prove the inverse-closedness of a subalgebra of $\mathcal{B}(\ell_w^q)$. Here are three of them: (i) the indirect approach, such as the Gelfand's technique [17, 19, 25]; (ii) the semi-direct approach, such as the bootstrap

argument [29] and the derivation trick [23]; (iii) the direct approach, such as the commutator trick [6, 39] and the asymptotic estimate technique [38, 41, 43]. Each approach has its advantages and limitations. For instance, the Gelfand technique and the asymptotic estimate technique work well for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q = 2$, but they are not directly applicable for inverse-closed subalgebras of $\mathcal{B}(\ell^q)$ with $q \neq 2$. The commutator trick is applicable to establish Wiener's lemma for subalgebra of $\mathcal{B}(\ell^q)$, $1 \leq q \leq \infty$ [6, 39]. In Section 5, we combine the commutator trick, the asymptotic estimate technique and the equivalence of ℓ_w^q -stability for different exponents q and for different discrete A_q -weights w , and then establish Wiener's lemma for subalgebras of $\mathcal{B}(\ell_w^q)$, where $1 \leq q < \infty$ and w is a discrete A_q -weight.

Theorem 1.4. *Let $1 \leq q < \infty$ and let w be a discrete A_q -weight. Then $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$.*

As an application of Theorem 1.4, we obtain Wiener's lemma for the Beurling algebra $A^*(\mathbb{T})$ of periodic functions [7].

Corollary 1.5. *If $f \in A^*(\mathbb{T})$ and $f(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ then $1/f \in A^*(\mathbb{T})$.*

As applications of Theorems 1.2 and 1.4, we establish the equivalence between the ℓ_w^q -stability of an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the existence of its left inverse in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.

Corollary 1.6. *Let $1 \leq q \leq \infty$, and let w be a discrete A_q -weight. Then an infinite matrix in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ has ℓ_w^q -stability if and only if it has a left inverse in $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$.*

2. A CLASS OF INFINITE MATRICES

In this section, we introduce a class of infinite matrices with off-diagonal decay, which includes the class $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$ in the Introduction as a special case.

A *weight matrix* on $\mathbb{Z}^d \times \mathbb{Z}^d$, or a *weight matrix* for brevity, is a positive matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$ with each entry not less than one, i.e.,

$$(2.1) \quad u(i, j) \geq 1 \quad \text{for all } i, j \in \mathbb{Z}^d.$$

For $1 \leq p \leq \infty$ and a weight matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$, define

$$(2.2) \quad \mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},$$

where

$$(2.3) \quad \|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)|u(i, j) \right)_{k \in \mathbb{Z}^d} \right\|_p.$$

If $p = 1$ and $u \equiv 1$ (i.e., all entries of the weight matrix u are equal to 1), then

$$(2.4) \quad \mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \quad \text{and} \quad \|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} = \|\cdot\|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)}.$$

In this paper, we use $\mathcal{B}_{p,u}$, \mathcal{B} , $\|\cdot\|_{\mathcal{B}_{p,u}}$, $\|\cdot\|_{\mathcal{B}}$ instead of $\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, $\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)$, $\|\cdot\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)}$, $\|\cdot\|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)}$ for brevity.

Remark 2.1. Let $1 \leq p \leq \infty$ and u be a weighted matrix. Define the *Gröchenig-Schur class* $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$(2.5) \quad \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i,j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},$$

where

$$(2.6) \quad \|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \max \left(\sup_{i \in \mathbb{Z}^d} \left\| (a(i, j)u(i, j))_{j \in \mathbb{Z}^d} \right\|_p, \sup_{j \in \mathbb{Z}^d} \left\| (a(i, j)u(i, j))_{i \in \mathbb{Z}^d} \right\|_p \right)$$

[25, 37, 41, 43]. For $p = 1$, the class $\mathcal{S}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ were introduced by Schur [37] for weight matrices $u := (w(i)/w(j))_{i,j \in \mathbb{Z}^d}$ generated by positive sequences $w := (w(i))_{i \in \mathbb{Z}^d}$, and by Gröchenig and Leinert [25] for weight matrices $u := (v(i-j))_{i,j \in \mathbb{Z}^d}$ associated with positive functions v on \mathbb{R}^d . For $1 \leq p \leq \infty$, the class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Sun for polynomial weights $u := ((1 + |i-j|_\infty)^\alpha)_{i,j \in \mathbb{Z}^d}$ with $\alpha > d(1 - 1/p)$ in [41] and for any weighted matrix u in [43]. From the above definition of the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, the following inclusion follows:

$$(2.7) \quad \mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$$

for any $1 \leq p \leq \infty$ and for any weight matrix u .

Remark 2.2. Let $1 \leq p \leq \infty$ and u be a weighted matrix. Define the *Gohberg-Baskakov-Sjöstrand class* $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices by

$$(2.8) \quad \mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i,j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},$$

where

$$(2.9) \quad \|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \left\| \left(\sup_{i-j=k} |a(i, j)|u(i, j) \right)_{k \in \mathbb{Z}^d} \right\|_p$$

[6, 20, 25, 26, 32, 39, 43]. For $p = 1$ and the trivial weight matrix u_0 (i.e., $u_0(i, j) = 1$ for all $i, j \in \mathbb{Z}^d$), the class $\mathcal{C}_{1,u_0}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced by Gohberg, Kaashoek, and Woerdeman [20] as a generalization of

the class of Toeplitz matrices associated with summable sequences. It was reintroduced by Sjöstrand [39] in considering algebra of pseudo-differential operators. For $p = 1$ and nontrivial weight matrices $u := (v(i - j))_{i,j \in \mathbb{Z}^d}$ associated with positive functions v on \mathbb{R}^d , the class $\mathcal{C}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced and studied by Baskakov [6] and Kurbatov [32] independently, see also [25]. The above definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ is given by Sun [43] for any $1 \leq p \leq \infty$ and any weight matrix u . From the definition of the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, we have the following inclusion:

$$(2.10) \quad \mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$$

for any $1 \leq p \leq \infty$ and for any weight matrix u .

Remark 2.3. The inclusions (2.7) and (2.10) become equalities for $p = \infty$, i.e.,

$$(2.11) \quad \mathcal{B}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{C}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{S}_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) =: \mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$$

The class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices is usually known as the *Jaffard class*, [6, 13, 23, 25, 29, 41, 43]. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with polynomial weight $u := ((1 + |i - j|)^\alpha)_{i,j \in \mathbb{Z}^d}$ was introduced by Jaffard [29] in considering wavelets on an open domain. The Jaffard class $\mathcal{J}_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with weight matrices $u := (v(i - j))_{i,j \in \mathbb{Z}^d}$ associated with positive functions v on \mathbb{R}^d was introduced by Baskakov [6] independently, and later applied nontrivially in the study of localization of frames [25], adaptive computation [13], and nonuniform sampling [42].

For the class $\mathcal{B}_{p,u}$ of infinite matrices, we have the following proposition.

Proposition 2.4. *Let $\alpha \in \mathbb{C}$, $1 \leq p \leq \infty$, u be a weight matrix, and let $A := ((a(i, j))_{i,j \in \mathbb{Z}^d}$ and $B := (b(i, j))_{i,j \in \mathbb{Z}^d}$ belong to $\mathcal{B}_{p,u}$. Then*

- (i) $\|A + B\|_{\mathcal{B}_{p,u}} \leq \|A\|_{\mathcal{B}_{p,u}} + \|B\|_{\mathcal{B}_{p,u}}$.
- (ii) $\|\alpha A\|_{\mathcal{B}_{p,u}} = |\alpha| \|A\|_{\mathcal{B}_{p,u}}$.
- (iii) $\|A^*\|_{\mathcal{B}_{p,u}} = \|A\|_{\mathcal{B}_{p,u}}$ where $A^* := (\overline{a(j, i)})_{i,j \in \mathbb{Z}^d}$ is the conjugate transpose of the matrix A .
- (iv) $\|A\|_{\mathcal{B}_{p,u}} \leq \|B\|_{\mathcal{B}_{p,u}}$ if $|A| \leq |B|$, i.e., $|a(i, j)| \leq |b(i, j)|$ for all $i, j \in \mathbb{Z}^d$.

All conclusions in the above proposition follow directly from (2.2) and (2.3). From the conclusions (i) and (ii) of the above proposition, we see that $\|\cdot\|_{\mathcal{B}_{p,u}}$ is a norm on the class $\mathcal{B}_{p,u}$ of infinite matrices. The properties in the conclusion (iv) is usually known as the *solidness* of the matrix norm $\|\cdot\|_{\mathcal{B}_{p,u}}$.

3. ALGEBRAIC PROPERTIES

In this section, we establish some algebraic properties for the class $\mathcal{B}_{p,u}$ of infinite matrices and give a proof of Theorem 1.1.

Let us first recall the concept of a p -submultiplicative weight matrix u [25, 43, 44]. For $1 \leq p \leq \infty$, a weight matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$ is said to be p -submultiplicative if there exists another weight matrix $v := (v(i, j))_{i, j \in \mathbb{Z}^d}$ such that

$$(3.1) \quad v(i, j) \geq 1 \quad \text{for all } i, j \in \mathbb{Z}^d,$$

$$(3.2) \quad u(i, j) \leq u(i, k)v(k, j) + v(i, k)u(k, j) \quad \text{for all } i, j, k \in \mathbb{Z}^d,$$

and

$$(3.3) \quad C_p(v, u) := \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty} (v(i, j)(u(i, j))^{-1}) \right) \right\|_{p/(p-1)} < \infty.$$

For $p = 1$, we simply say that a weight matrix is *submultiplicative* instead of 1-submultiplicative. We call the weight matrix v satisfying (3.1), (3.2) and (3.3) a *companion weight matrix* of the p -submultiplicative weight matrix u . Denote by $C(u)$ the set of all companion weights of a p -submultiplicative weight matrix u , and define the p -submultiplicative bound $M_p(u)$ by

$$(3.4) \quad M_p(u) := \inf_{v \in C(u)} C_p(v, u).$$

One may verify that $C(u)$ is a convex set and the infimum of $C_p(v, u)$ in the set $C(u)$ can be attained for some companion weight matrix v . So from now on, except stated explicitly, we **always** assume that the companion weight v of a p -submultiplicative weight matrix u is the one satisfying

$$(3.5) \quad M_p(u) = C_p(v, u).$$

Remark 3.1. From the definitions of p -submultiplicative weight matrices on $\mathbb{Z}^d \times \mathbb{Z}^d$, we have the following:

- (i) A p -submultiplicative weight matrix is q -submultiplicative for all $1 \leq q \leq p$.
- (ii) A necessary condition for a weight matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$ to be p -submultiplicative is $u(i, j) \leq Cu(i, k)u(k, j)$ for all $i, j, k \in \mathbb{Z}^d$ and for some positive constant C . When $p = 1$, the above necessary condition is also a sufficient condition [25].
- (iii) Let $1 \leq p \leq \infty, \delta \in (0, 1)$, and let α be a number with the property that $\alpha > d - d/p$ if $1 < p \leq \infty$, and $\alpha \geq 0$ if $p = 1$. Then the Toeplitz matrices $p_\alpha := ((1 + |i - j|_\infty)^\alpha)_{i, j \in \mathbb{Z}^d}$

generated by the polynomial weight $(1 + |x|_\infty)^\alpha$, and $e_\delta := (\exp(|i - j|_\infty^\delta))_{i,j \in \mathbb{Z}^d}$ generated by the sub-exponential weight $\exp(|x|_\infty^\delta)$, are p -submultiplicative [43].

Now we state the main result of this section, an extension of Theorem 1.1.

Theorem 3.2. *Let $1 \leq p, q \leq \infty$, u be a p -submultiplicative weight matrix with the p -submultiplicative bound $M_p(u)$, and let w be a discrete A_q -weight with the A_q -bound $A_q(w)$. Then the following statements hold.*

(i) *If v is a companion weight matrix of the p -submultiplicative weight matrix u , then*

$$(3.6) \quad \|AB\|_{\mathcal{B}_{p,u}} \leq 2^{2/p} 5^{(d-1)/p} (\|A\|_{\mathcal{B}_{p,u}} \|B\|_{\mathcal{B}_{1,v}} + \|A\|_{\mathcal{B}_{1,v}} \|B\|_{\mathcal{B}_{p,u}})$$

for all $A, B \in \mathcal{B}_{p,u}$.

(ii) *$\mathcal{B}_{p,u}$ is (and hence \mathcal{B} is also) an algebra. Moreover*

$$(3.7) \quad \|AB\|_{\mathcal{B}_{p,u}} \leq 2^{1+2/p} 5^{(d-1)/p} M_p(u) \|A\|_{\mathcal{B}_{p,u}} \|B\|_{\mathcal{B}_{p,u}} \quad \text{for all } A, B \in \mathcal{B}_{p,u}.$$

(iii) *$\mathcal{B}_{p,u}$ is a subalgebra of \mathcal{B} . Moreover*

$$(3.8) \quad \|A\|_{\mathcal{B}} \leq M_p(u) \|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}.$$

(iv) *$\mathcal{B}_{p,u}$ is (and hence \mathcal{B} is also) a subalgebra of $\mathcal{B}(\ell_w^q)$. Moreover*

$$(3.9) \quad \|Ac\|_{q,w} \leq 2^{2d} 3^{d/q} (A_q(w))^{1/q} M_p(u) \|A\|_{\mathcal{B}_{p,u}} \|c\|_{q,w}$$

for all $A \in \mathcal{B}_{p,u}$ and $c \in \ell_w^q$.

Before we give the proof of the above theorem, let us next have some remarks on the unital Banach algebra property of the algebra $\mathcal{B}_{p,u}$, on the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$, and on the inclusion $\mathcal{B}_{p,u} \subset \mathcal{B}(\ell_w^q)$.

Remark 3.3. For $1 \leq p \leq \infty$ and a p -submultiplicative weight matrix $u := (u(i, j))_{i,j \in \mathbb{Z}^d}$, following the standard procedure [19, 35] we define $\|A\|'_{\mathcal{B}_{p,u}} := \sup_{\|B\|_{\mathcal{B}_{p,u}}=1} \|AB\|_{\mathcal{B}_{p,u}}$ for $A \in \mathcal{B}_{p,u}$. Then

$$(3.10) \quad \|AB\|'_{\mathcal{B}_{p,u}} \leq \|A\|'_{\mathcal{B}_{p,u}} \|B\|'_{\mathcal{B}_{p,u}} \quad \text{for all } A, B \in \mathcal{B}_{p,u}.$$

If the weight matrix u further satisfies

$$(3.11) \quad M := \sup_{i \in \mathbb{Z}^d} u(i, i) < \infty,$$

then the identity matrix I belongs to $\mathcal{B}_{p,u}$, and the norms $\|\cdot\|_{\mathcal{B}_{p,u}}$ and $\|\cdot\|'_{\mathcal{B}_{p,u}}$ on $\mathcal{B}_{p,u}$ are equivalent to each other, because

$$M^{-1} \|A\|_{\mathcal{B}_{p,u}} \leq \|A\|'_{\mathcal{B}_{p,u}} \leq 2^{1+2/p} 5^{(d-1)/p} M_p(u) \|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}$$

by the conclusion (ii) of Theorem 3.2 and the fact that $\|I\|_{\mathcal{B}_{p,u}} = M$. Therefore if $1 \leq p \leq \infty$ and u is a p -submultiplicative weight matrix satisfying (3.11), then the class $\mathcal{B}_{p,u}$ of infinite matrices endowed with the norm $\|\cdot\|'_{\mathcal{B}_{p,u}}$ becomes a unital Banach algebra.

Remark 3.4. Let $1 \leq p \leq \infty$, u be a p -submultiplicative weight matrix, and v be its companion weight matrix. If the companion weight matrix v is submultiplicative, then both $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ are algebras by the conclusion (ii) of Theorem 3.2, and $\mathcal{B}_{p,u}$ is a subalgebra of $\mathcal{B}_{1,v}$ since

$$(3.12) \quad \|A\|_{\mathcal{B}_{1,v}} \leq C_p(v, u) \|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u}.$$

Applying (3.12) with A replaced by A^n and then taking n -th roots and the limit as $n \rightarrow \infty$ yields

$$\rho_{\mathcal{B}_{1,v}}(A) := \limsup_{n \rightarrow \infty} (\|A^n\|_{\mathcal{B}_{1,v}})^{1/n} \leq \limsup_{n \rightarrow \infty} (\|A^n\|_{\mathcal{B}_{p,u}})^{1/n} =: \rho_{\mathcal{B}_{p,u}}(A).$$

Applying the conclusion (i) of Theorem 3.2 gives

$$(3.13) \quad \|A^{2n}\|_{\mathcal{B}_{p,u}} \leq 2^{1+2/p} 5^{(d-1)/p} \|A^n\|_{\mathcal{B}_{p,u}} \|A^n\|_{\mathcal{B}_{1,v}},$$

and then taking n -th roots and letting $n \rightarrow \infty$ lead to the inequality $\rho_{\mathcal{B}_{p,u}}(A) \leq \rho_{\mathcal{B}_{1,v}}(A)$. This concludes that if u is a p -submultiplicative weight matrix and its companion weight matrix v is submultiplicative, then the spectral radii $\rho_{\mathcal{B}_{p,u}}(A)$ and $\rho_{\mathcal{B}_{1,v}}(A)$ are the same for any $A \in \mathcal{B}_{p,u}$, i.e., $\rho_{\mathcal{B}_{1,v}}(A) = \rho_{\mathcal{B}_{p,u}}(A)$ for all $A \in \mathcal{B}_{p,u}$. The above procedure to establish the equality of spectral radii in the algebras $\mathcal{B}_{p,u}$ and $\mathcal{B}_{1,v}$ from the inequality in the conclusion (i) of Theorem 3.2 is known as *Brandenburg's trick* [11, 23]. Another technique to prove the equality of spectral radii in two algebras \mathcal{A}_1 and \mathcal{A}_2 with the same identity is by showing that

$$(3.14) \quad \|A\|_{\mathcal{A}_2} \leq C \|A\|_{\mathcal{A}_1}$$

and

$$(3.15) \quad \|A^2\|_{\mathcal{A}_1} \leq C \|A\|_{\mathcal{A}_1}^{1+\theta} \|A\|_{\mathcal{A}_2}^{1-\theta} \quad \text{for all } A \in \mathcal{A}_1,$$

where $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$ are norms in the algebra \mathcal{A}_1 and \mathcal{A}_2 respectively, and where $C \in (0, \infty)$ and $\theta \in [0, 1)$ are constants independent of $A \in \mathcal{A}$. The estimates in (3.14) and (3.15) for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ or $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ are established in [41, 43], while the ones for $\mathcal{A}_2 = \mathcal{B}(\ell^2)$ and $\mathcal{A}_1 = \mathcal{B}_{p,u}$, $1 \leq p \leq \infty$, are given in Lemma 5.3.

Remark 3.5. The conclusion (iv) of Theorem 3.2 about the boundedness of an infinite matrix in \mathcal{B} on the weight sequence space ℓ_w^q is a simplified discrete version of the second conclusion in [40, Proposition 2 of Chapter 10]. The reader may refer to [26, Lemma 3.1] for a general result on the boundedness of an infinite matrix on sequence spaces.

We conclude this section by giving the proof of Theorem 3.2.

Proof of Theorem 3.2. (i): Let $1 \leq p \leq \infty$, u be a p -submultiplicative weight matrix, and let v be a companion weight matrix of the weight matrix u . Take $A := (a(i, j))_{i, j \in \mathbb{Z}^d}$ and $B := (b(i, j))_{i, j \in \mathbb{Z}^d}$ in $\mathcal{B}_{p, u}$, and write $AB := (c(i, j))_{i, j \in \mathbb{Z}^d}$. Then it follows from (3.2) that

$$\begin{aligned}
 |c(i, j)|u(i, j) &= \left| \sum_{k \in \mathbb{Z}^d} a(i, k)b(k, j) \right| u(i, j) \\
 &\leq \sum_{k \in \mathbb{Z}^d} |a(i, k)|u(i, k)|b(k, j)|v(k, j) \\
 (3.16) \quad &+ \sum_{k \in \mathbb{Z}^d} |a(i, k)|v(i, k)|b(k, j)|u(k, j) \quad \text{for all } i, j \in \mathbb{Z}^d.
 \end{aligned}$$

For $1 \leq p < \infty$, we obtain from (3.16) that

$$\begin{aligned}
 &\sum_{k \in \mathbb{Z}^d} |a(i, k)|u(i, k)|b(k, j)|v(k, j) \\
 &\leq \left(\sum_{k' \in \mathbb{Z}^d} (|a(i, k')|u(i, k'))^p |b(k', j)|v(k', j)) \right)^{1/p} \\
 &\quad \times \left(\sum_{k'' \in \mathbb{Z}^d} |b(k'', j)|v(k'', j) \right)^{(p-1)/p} \\
 &\leq (\|B\|_{\mathcal{B}_{1, v}})^{(p-1)/p} \left\{ \left(\sum_{k' \in \mathbb{Z}^d} |b(k', j)|v(k', j) \right) \right. \\
 &\quad \times \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|a(i', j')|u(i', j'))^p \right) \\
 &\quad + \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|b(i', j')|v(i', j')) \right) \\
 &\quad \left. \times \left(\sum_{k' \in \mathbb{Z}^d} (|a(i, k')|u(i, k'))^p \right) \right\}^{1/p} \\
 &\leq (\|B\|_{\mathcal{B}_{1, v}})^{(p-1)/p} \left\{ \|B\|_{\mathcal{B}_{1, v}} \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|a(i', j')|u(i', j'))^p \right) \right. \\
 &\quad \left. + (\|A\|_{\mathcal{B}_{p, u}})^p \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|b(i', j')|v(i', j')) \right) \right\}^{1/p},
 \end{aligned}$$

and

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^d} |a(i, k)| |v(i, k)| |b(k, j)| |u(k, j)| \\
& \leq (\|A\|_{\mathcal{B}_{1,v}})^{(p-1)/p} \left\{ \|A\|_{\mathcal{B}_{1,v}} \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|b(i', j')| |u(i', j')|)^p \right) \right. \\
& \quad \left. + (\|B\|_{\mathcal{B}_{p,u}})^p \left(\sup_{|i'-j'|_\infty \geq |i-j|_\infty/2} (|a(i', j')| |v(i', j')|) \right) \right\}^{1/p}.
\end{aligned}$$

Combining the above two estimates with (3.16) leads to

$$\begin{aligned}
\|AB\|_{\mathcal{B}_{p,u}} &= \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty} (|c(i, j)| |u(i, j)|) \right)_{k \in \mathbb{Z}^d} \right\|_p \\
&\leq (\|B\|_{\mathcal{B}_{1,v}})^{(p-1)/p} \left\{ \|B\|_{\mathcal{B}_{1,v}} \right. \\
&\quad \times \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty/2} (|a(i', j')| |u(i', j')|)^p \right)_{k \in \mathbb{Z}^d} \right\|_1 \\
&\quad + (\|A\|_{\mathcal{B}_{p,u}})^p \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty/2} (|b(i', j')| |v(i', j')|) \right)_{k \in \mathbb{Z}^d} \right\|_1 \Big\}^{1/p} \\
&\quad + (\|A\|_{\mathcal{B}_{1,v}})^{(p-1)/p} \left\{ \|A\|_{\mathcal{B}_{1,v}} \right. \\
&\quad \times \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty/2} (|b(i', j')| |u(i', j')|)^p \right)_{k \in \mathbb{Z}^d} \right\|_1 \\
&\quad + (\|B\|_{\mathcal{B}_{p,u}})^p \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty/2} (|a(i', j')| |v(i', j')|) \right)_{k \in \mathbb{Z}^d} \right\|_1 \Big\}^{1/p} \\
&\leq 2^{2/p} 5^{(d-1)/p} \left(\|A\|_{\mathcal{B}_{p,u}} \|B\|_{\mathcal{B}_{1,v}} + \|A\|_{\mathcal{B}_{1,v}} \|B\|_{\mathcal{B}_{p,u}} \right),
\end{aligned}$$

where we have used the fact that

$$(3.17) \quad \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty/N} |a(i, j)| \right) \right\|_1 \leq N(2N+1)^{d-1} \left\| \left(\sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \right) \right\|_1$$

for any integer $N \geq 1$ and $A := (a(i, j)) \in \mathcal{B}$. This proves (3.6) for $1 \leq p < \infty$.

For $p = \infty$, it follows from (3.16) that

$$(3.18) \quad \|AB\|_{\mathcal{B}_{\infty,u}} \leq \|A\|_{\mathcal{B}_{\infty,u}} \|B\|_{\mathcal{B}_{1,v}} + \|A\|_{\mathcal{B}_{1,v}} \|B\|_{\mathcal{B}_{\infty,u}}.$$

Hence (3.6) for $p = \infty$ is proved.

(ii) Let v be the companion weight matrix of the p -submultiplicative weight u that satisfies (3.1)–(3.3) and (3.5). Then

$$(3.19) \quad \|A\|_{\mathcal{B}_{1,v}} \leq M_p(u) \|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u},$$

because

$$(3.20) \quad \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)|v(i, j) \leq \left(\sup_{|i'-j'|_\infty \geq |k|_\infty} |a(i', j')|u(i', j') \right) \times \left(\sup_{|i'-j'|_\infty \geq |k|_\infty} v(i', j')(u(i', j'))^{-1} \right)$$

hold for all $k \in \mathbb{Z}^d$. Combining (3.6) and (3.19) proves (3.7).

(iii) Let v be the companion weight matrix of the p -submultiplicative weight u that satisfies (3.1)–(3.3) and (3.5). Then

$$(3.21) \quad \|A\|_{\mathcal{B}} \leq \|A\|_{\mathcal{B}_{1,v}} \quad \text{for all } A \in \mathcal{B}_{1,v}$$

by (3.1) for the weight matrix v . This together with (3.19) gives (3.8) and hence proves the conclusion (iii).

(iv) By (iii), it suffices to prove

$$(3.22) \quad \|Ac\|_{q,w} \leq 2^{2d} 3^{d/q} (A_q(w))^{1/q} \|A\|_{\mathcal{B}} \|c\|_{q,w}$$

for all $A := (a(i, j))_{i,j \in \mathbb{Z}^d} \in \mathcal{B}$ and $c \in \ell_w^q$. Set $h(n) := \sup_{|i-j|_\infty \geq n} |a(i, j)|$. Then $\{h(n)\}_{n=0}^\infty$ is a decreasing sequence, i.e., $h(n+1) \leq h(n)$ for all $n \geq 0$, and

$$(3.23) \quad \begin{aligned} \sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} &\leq 2^{2d} h(1) + 2^{d+2} \sum_{l=2}^\infty \left(\sum_{2^{l-2} < s \leq 2^{l-1}} h(s) \right) 2^{l(d-1)} \\ &\leq 2^{2d} h(1) + 2^{3d} \sum_{s=2}^\infty h(s) s^{d-1} \\ &\leq 2^{2d} h(1) + 2^{2d} d^{-1} \sum_{s=2}^\infty \sum_{k \in \mathbb{Z}^d \text{ with } |k|_\infty = s} h(|k|_\infty) \\ &\leq 2^{2d} (\|A\|_{\mathcal{B}} - h(0)). \end{aligned}$$

For $1 < q < \infty$ and a discrete A_q -weight w ,

$$\begin{aligned} \|Ac\|_{q,w} &\leq \left\{ \sum_{i \in \mathbb{Z}^d} \left(\sum_{j \in \mathbb{Z}^d} h(|i-j|_\infty) |c(j)| \right)^q w(i) \right\}^{1/q} \\ &\leq h(0) \left\{ \sum_{i \in \mathbb{Z}^d} |c(i)|^q w(i) \right\}^{1/q} + \left\{ \sum_{i \in \mathbb{Z}^d} w(i) \left(\sum_{l=1}^\infty h(2^{l-1}) 2^{(l+1)d} \right)^{q-1} \right. \\ &\quad \left. \times \left(\sum_{l=1}^\infty h(2^{l-1}) 2^{-(l+1)d(q-1)} \left(\sum_{2^{l-1} \leq |i-j|_\infty < 2^l} |c(j)| \right)^q \right) \right\}^{1/q}. \end{aligned}$$

Thus

$$\begin{aligned}
\|Ac\|_{q,w} &\leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q} \\
&\quad \times \left\{ \sum_{i \in \mathbb{Z}^d} \sum_{l=1}^{\infty} w(i) h(2^{l-1}) 2^{-(l+1)d(q-1)} \right. \\
&\quad \times \left(\sum_{2^{l-1} \leq |i-j|_{\infty} < 2^l} |c(j)|^q w(j) \right) \\
&\quad \times \left(\sum_{2^{l-1} \leq |i-j'|_{\infty} < 2^l} (w(j'))^{-1/(q-1)} \right)^{q-1} \Big\}^{1/q}.
\end{aligned}$$

This together with the discrete A_q -weight assumption leads to

$$\begin{aligned}
\|Ac\|_{q,w} &\leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q} (A_q(w))^{1/q} \\
&\quad \times \left\{ \sum_{l=1}^{\infty} h(2^{l-1}) 2^{(l+1)d} \sum_{i \in \mathbb{Z}^d} \frac{w(i)}{\sum_{|i-j'|_{\infty} < 2^l} w(j')} \right. \\
&\quad \times \left(\sum_{2^{l-1} \leq |i-j|_{\infty} < 2^l} |c(j)|^q w(j) \right) \Big\}^{1/q} \\
&\leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{\mathcal{B}} - h(0))^{(q-1)/q} (A_q(w))^{1/q} \\
&\quad \times \left\{ \sum_{l=1}^{\infty} h(2^{l-1}) 2^{(l+1)d} \left(\sum_{j \in \mathbb{Z}^d} |c(j)|^q w(j) \right. \right. \\
&\quad \left. \left(\sum_{\epsilon \in \{-1,0,1\}^d} \sum_{|i-j-\epsilon 2^{l-1}|_{\infty} < 2^{l-1}} \frac{w(i)}{\sum_{|i-j'|_{\infty} < 2^l} w(j')} \right) \right) \Big\}^{1/q} \\
&\leq 2^{2d} 3^{d/p} (A_q(w))^{1/q} \|A\|_{\mathcal{B}} \|c\|_{q,w},
\end{aligned}$$

and hence (3.9) for $1 < q < \infty$ is established.

The conclusion (3.9) for $q = 1$ can be proved by similar arguments. We omit the details here. \square

4. ℓ_w^q -STABILITY

In this section, we prove the following theorem, a slight generalization of Theorem 1.2, and Corollary 1.3. We also provide a characterization to the ℓ_w^q -stability of a Toeplitz matrix in \mathcal{B} .

Theorem 4.1. *Let $1 \leq p \leq \infty$, $1 \leq q, q' < \infty$, let the weight matrix u be p -submultiplicative, and w, w' be discrete A_q -weight and $A_{q'}$ -weight respectively. If $A \in \mathcal{B}_{p,u}$ has ℓ_w^q -stability, then A has $\ell_{w'}^{q'}$ -stability.*

As the trivial weight w_0 (i.e. $w_0(i) = 1$ for all $i \in \mathbb{Z}^d$) is a discrete A_q -weight for any $1 \leq q < \infty$, we have the following corollary of Theorem 4.1. Similar result about ℓ^q -stability for different exponents $q \in [1, \infty]$ is established by Aldroubi, Baskakov and Krishtal [1] for infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{1,p_\alpha}(\mathbb{Z}^d, \mathbb{Z}^d)$ with $\alpha > (d+1)^2$, by Tessera [49] for $\alpha > 0$, and by Shin and Sun [38] for $\alpha \geq 0$, where $p_\alpha = ((1 + |i - j|_\infty)^\alpha)_{i,j \in \mathbb{Z}^d}$.

Corollary 4.2. *If $A \in \mathcal{B}$ has ℓ^q -stability for some $1 \leq q < \infty$, then A has $\ell^{q'}$ -stability for all $1 \leq q' < \infty$.*

The ℓ_w^q -stability is one of the basic assumptions for infinite matrices arising in many fields of mathematics (see [2, 4, 24, 28, 29, 39, 42, 43] for a sample of papers), but little is known about practical criteria for the ℓ_w^q -stability of an infinite matrix, see [46] for the diagonal-blocks-dominated criterion for the ℓ^2 -stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$. As an application of Theorem 1.2, we have the following characterization to the ℓ_w^q -stability of a Toeplitz matrix in \mathcal{B} .

Corollary 4.3. *Let $1 \leq q < \infty$, $A := (a(i - j))_{i,j \in \mathbb{Z}^d}$ be a Toeplitz matrix in \mathcal{B} , and let w be a discrete A_q -weight. Then A has ℓ_w^q -stability if and only if $\hat{a}(\xi) := \sum_{n \in \mathbb{Z}^d} a(n) e^{-\sqrt{-1} n \xi} \neq 0$ for all $\xi \in \mathbb{R}^d$.*

To prove Theorem 4.1, we recall a characterization for discrete A_q -weights.

Lemma 4.4. [18, 40] *Let $1 \leq q < \infty$. Then $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete A_q -weight with the A_q -bound $A_q(w)$ if and only if*

$$(4.1) \quad \left(N^{-d} \sum_{i \in a + [0, N-1]^d} |c(i)| \right)^q \left(N^{-d} \sum_{i \in a + [0, N-1]^d} w(i) \right) \leq A_q(w) N^{-d} \sum_{i \in a + [0, N-1]^d} |c(i)|^q w(i)$$

hold for all $a \in \mathbb{Z}^d, 1 \leq N \in \mathbb{Z}$ and sequences $c := (c(i))_{i \in \mathbb{Z}^d}$.

To prove Theorem 4.1, we need a technical lemma about estimating a bounded sequence c via the sequence Ac , which will also be used later in the proof of Theorem 1.4. Similar estimate is given in [39] when the infinite matrix A belongs to the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}(\mathbb{Z}^d, \mathbb{Z}^d)$ and has ℓ_w^p -stability for the trivial weight $w \equiv 1$.

Lemma 4.5. *Let $1 \leq q < \infty$, and w be a discrete A_q -weight. If $A \in \mathcal{B}$ has ℓ_w^q -stability, then there exists a nonnegative sequence $\{g(i)\}_{i \in \mathbb{Z}^d}$ on*

\mathbb{Z}^d such that

$$(4.2) \quad \sum_{k \in \mathbb{Z}^d} \left(\sup_{|i|_\infty \geq |k|_\infty} g(i) \right) < \infty$$

and

$$(4.3) \quad |c(i)| \leq \sum_{j \in \mathbb{Z}^d} g(i-j) |(Ac)(j)|, \quad i \in \mathbb{Z}^d,$$

where $c \in \ell^\infty$.

Proof. Without loss of generality, we assume that

$$(4.4) \quad \|c\|_{q,w} \leq \|Ac\|_{q,w} \quad \text{for all } c \in \ell_w^q.$$

Let $h(x) = \min(\max(2-|x|_\infty, 0), 1)$ and N be a sufficiently large integer chosen later. Define linear operators $\Psi_n^N, n \in N\mathbb{Z}^d$, on ℓ_w^q by

$$(4.5) \quad \Psi_n^N c := \left(h\left(\frac{j-n}{N}\right) c(j) \right)_{j \in \mathbb{Z}^d} \quad \text{for } c := (c(j))_{j \in \mathbb{Z}^d} \in \ell_w^q.$$

Then for $c := (c(j))_{j \in \mathbb{Z}^d} \in \ell_w^q$ and $|n - n'|_\infty \leq 8N$,

$$\begin{aligned} & \|(\Psi_n^N A - A \Psi_{n'}^N) \Psi_{n'}^N c\|_{q,w} \\ &= \left\{ \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \mathbb{Z}^d} \left(h\left(\frac{i-n}{N}\right) - h\left(\frac{j-n'}{N}\right) \right) \right. \right. \\ & \quad \left. \left. \times a(i, j) h\left(\frac{j-n'}{N}\right) c(j) \right|^q w(i) \right\}^{1/q} \\ &\leq N^{-1/2} \left\{ \sum_{i \in \mathbb{Z}^d} \left(\sum_{|i-j|_\infty \leq \sqrt{N}} |a(i, j)| |c(j)| \right)^q w(i) \right\}^{1/q} \\ & \quad + \left\{ \sum_{i \in \mathbb{Z}^d} \left(\sum_{|i-j|_\infty > \sqrt{N}} |a(i, j)| |c(j)| \right)^q w(i) \right\}^{1/q} \\ &\leq \left\{ 2^{2d+2d/q} N^{-1/2} (A_q(w))^{1/q} \|A\|_{\mathcal{B}} + 2^{3d+2d/q+1} (A_q(w))^{1/q} \right. \\ (4.6) \quad & \left. \times \left(\sum_{|k|_\infty \geq \sqrt{N}/2} \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \right) \right\} \|c\|_{q,w}, \end{aligned}$$

where the last inequality follows from Theorem 3.2 and the following estimate:

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^d} \sup_{|i-j|_\infty \geq \max(|k|_\infty, \sqrt{N})} |a(i, j)| \\
& \leq (2\sqrt{N} + 1)^d \sup_{|i-j|_\infty \geq \sqrt{N}} |a(i, j)| + \sum_{|k|_\infty > \sqrt{N}} \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \\
& \leq 2^{d+1} \sum_{|k|_\infty \geq \sqrt{N}/2} \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)|.
\end{aligned}$$

Similarly for $c := (c(j))_{j \in \mathbb{Z}^d} \in \ell_w^q$ and $|n - n'|_\infty > 8N$,

$$\begin{aligned}
& \|(\Psi_n^N A - A \Psi_n^N) \Psi_{n'}^N c\|_{q,w} \\
& = \left(\sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \mathbb{Z}^d} h\left(\frac{i-n}{N}\right) a(i, j) h\left(\frac{j-n'}{N}\right) c(j) \right|^q w(i) \right)^{1/q} \\
& \leq \left(\sup_{|i'-j'|_\infty \geq |n-n'|_\infty/2} |a(i', j')| \right) \left(\sum_{|i-n|_\infty < 2N} \left(\sum_{|j-n'|_\infty < 2N} |c(j)| \right)^q w(i) \right)^{1/q} \\
& \leq 2^{2d} N^d (A_q(w))^{1/q} \left(\sup_{|i'-j'|_\infty \geq |n-n'|_\infty/2} |a(i', j')| \right) \\
& \quad \times \left(\frac{\sum_{|i'-n|_\infty < 2N} w(i')}{\sum_{|i'-n'|_\infty < 2N} w(i')} \right)^{1/q} \|c\|_{q,w}.
\end{aligned} \tag{4.7}$$

Define

$$\alpha_n := \sum_{|i'-n|_\infty < 2N} w(i'), \quad n \in N\mathbb{Z}^d. \tag{4.8}$$

and the linear operator Φ_N on ℓ_w^p by

$$\Phi_N c := \left(\left(\sum_{n \in N\mathbb{Z}^d} \left(h\left(\frac{j-n}{N}\right) \right)^2 \right)^{-1} c(j) \right)_{j \in \mathbb{Z}^d} \text{ for } c := (c(j))_{j \in \mathbb{Z}^d} \in \ell_w^p. \tag{4.9}$$

Then for all $n' \in N\mathbb{Z}^d$ with $|n - n'| \leq 8N$,

$$\alpha_n \leq \sum_{|i'-n'|_\infty < 10N} w(i') \leq 6^{dq} A_q(w) \alpha_{n'} \tag{4.10}$$

by (4.1), and

$$\|\Phi_N c\|_{q,w} \leq \|c\|_{q,w} \quad \text{for all } c \in \ell_w^q. \tag{4.11}$$

Note that $\Psi_n^N c \in \ell_w^p$ for any $c \in \ell^\infty$ and $n \in N\mathbb{Z}^d$, and

$$\|\Psi_n^N c\|_{q,w} \leq \alpha_n^{1/q} \|c\|_\infty, \quad n \in N\mathbb{Z}^d. \tag{4.12}$$

Then for $c \in \ell^\infty$, combining (4.4), (4.6), (4.7), (4.10), and (4.11) leads to

$$\begin{aligned}
(4.13) \quad & \alpha_n^{-1/q} \|\Psi_n^N c\|_{q,w} \leq \alpha_n^{-1/q} \|A \Psi_n^N c\|_{q,w} \\
& \leq \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} + \alpha_n^{-1/q} \|(\Psi_n^N A - A \Psi_n^N) c\|_{q,w} \\
& \leq \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} + \alpha_n^{-1/q} \sum_{n' \in N\mathbb{Z}^d} \|(\Psi_n^N A - A \Psi_n^N) \Psi_{n'}^N \Phi_N \Psi_{n'}^N c\|_{q,w} \\
& \leq \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} + 2^{2d+2d/q} 6^d (A_q(w))^{2/q} \sum_{|n'-n|_\infty \leq 8N} \alpha_{n'}^{-1/q} \|\Psi_{n'}^N c\|_{q,w} \\
& \quad \times \left\{ \left(N^{-1/2} \|A\|_{\mathcal{B}} + 2^{d+1} \sum_{|k|_\infty \geq \sqrt{N}/2} \sup_{|i'-j'|_\infty \geq |k|_\infty} |a(i', j')| \right) \right\} \\
& \quad + \sum_{|n'-n|_\infty > 8N} \alpha_{n'}^{-1/q} \|\Psi_{n'}^N c\|_{q,w} \\
& \quad \times \left\{ 2^{2d} N^d (A_q(w))^{1/q} \left(\sup_{|i'-j'|_\infty \geq |n-n'|_\infty/2} |a(i', j')| \right) \right\} \\
& =: \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} + \sum_{n' \in N\mathbb{Z}^d} V_N(n - n') \alpha_{n'}^{-1/q} \|\Psi_{n'}^N c\|_{q,w}.
\end{aligned}$$

Define sequences $V_N^l := (V_N^l(n))_{n \in N\mathbb{Z}^d}$, $l \geq 1$, as follows:

$$(4.14) \quad \begin{cases} V_N^l(n) := V_N(n) & \text{if } l = 1 \text{ and } n \in N\mathbb{Z}^d, \\ V_N^l(n) := \sum_{n' \in N\mathbb{Z}^d} V_N(n - n') V_N^{l-1}(n') & \text{if } l \geq 2 \text{ and } n \in N\mathbb{Z}^d. \end{cases}$$

Then for $c \in \ell^\infty$, applying (4.13) repeatedly yields

$$\begin{aligned}
& \alpha_n^{-1/q} \|\Psi_n^N c\|_{q,w} \\
& \leq \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} + \sum_{l=1}^{l_0} \sum_{n' \in N\mathbb{Z}^d} V_N^l(n - n') \alpha_{n'}^{-1/q} \|\Psi_{n'}^N A c\|_{q,w} \\
(4.15) \quad & + \sum_{n' \in N\mathbb{Z}^d} V_N^{l_0+1}(n - n') \alpha_{n'}^{-1/q} \|\Psi_{n'}^N c\|_{q,w}, \quad l_0 \geq 1.
\end{aligned}$$

Set

$$(4.16) \quad \epsilon_N^l := \sum_{k \in N\mathbb{Z}^d} \sup_{|n|_\infty \geq |k|_\infty} |V_N^l(n)|.$$

Inductively for $l \geq 2$,

$$\begin{aligned}
\epsilon_N^l & \leq \epsilon_N^{l-1} \sum_{k \in N\mathbb{Z}^d} \sup_{|n|_\infty \geq |k|_\infty/2} |V_N(n)| \\
& \quad + \epsilon_N^1 \sum_{k \in N\mathbb{Z}^d} \sup_{|n|_\infty \geq |k|_\infty/2} |V_N^{l-1}(n)| \leq 5^d \epsilon_N^1 \epsilon_N^{l-1},
\end{aligned}$$

where we have used (3.17) to obtain the last inequality. This shows that

$$(4.17) \quad \epsilon_N^l \leq (5^d \epsilon_N^1)^l \quad \text{for all } l \geq 1.$$

Note that

$$\begin{aligned}
\epsilon_N^1 &\leq 2^{4d+2d/q} 3^{3d} (A_q(w))^{2/q} \left\{ N^d \left(\sup_{|i'-j'|_\infty > 4N} |a(i', j')| \right) \right. \\
&\quad + 2^{d+1} \left(\sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} \sup_{|i'-j'|_\infty \geq |k'|_\infty} |a(i', j')| \right) \\
&\quad \left. + N^{-1/2} \|A\|_{\mathcal{B}} \right\} + 2^{2d} (A_q(w))^{1/q} \\
&\quad \times \left\{ \sum_{|k|_\infty > 8N, k \in N\mathbb{Z}^d} N^d \left(\sup_{|i'-j'|_\infty \geq |k|_\infty/2} |a(i', j')| \right) \right\} \\
&\leq 2^{4d+2d/q} 3^{3d} (A_q(w))^{2/q} \left\{ N^{-1/2} \|A\|_{\mathcal{B}} \right. \\
&\quad + 2^{d+2} \left(\sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} \sup_{|i'-j'|_\infty \geq |k'|_\infty} |a(i', j')| \right) \\
&\quad \left. + 2^{2d+1} (A_q(w))^{1/q} \sum_{k' \in \mathbb{Z}^d, |k'| > 7N} \left(\sup_{|i'-j'|_\infty \geq 7|k'|_\infty/16} |a(i', j')| \right) \right\} \\
&\leq 2^{6d} 3^{3d} (A_q(w))^{2/q} N^{-1/2} \|A\|_{\mathcal{B}} + 2^{7d+3} 3^{3d} (A_q(w))^{2/q} \\
&\quad \times \left(\sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} \left(\sup_{|i'-j'|_\infty \geq |k'|_\infty} |a(i', j')| \right) \right) \\
&\rightarrow 0 \quad \text{as } N \rightarrow +\infty
\end{aligned}$$

by the assumption $A \in \mathcal{B}$. Let N be the integer chosen sufficiently large so that

$$(4.18) \quad \epsilon_N^1 < 5^{-d}.$$

Taking the limit as $l_0 \rightarrow \infty$ in (4.15), and using (4.12), (4.17) and (4.18) lead to

$$\begin{aligned}
\alpha_n^{-1/q} \|\Psi_n^N c\|_{q,w} &\leq \alpha_n^{-1/q} \|\Psi_n^N A c\|_{q,w} \\
&\quad + \sum_{n' \in N\mathbb{Z}^d} \left(\sum_{l=1}^{\infty} V_N^l(n-n') \right) \alpha_{n'}^{-1/q} \|\Psi_{n'}^N A c\|_{q,w} \\
(4.19) \quad &=: \sum_{n' \in N\mathbb{Z}^d} W_N(n-n') \alpha_{n'}^{-1/q} \|\Psi_{n'}^N A c\|_{q,w},
\end{aligned}$$

and

$$(4.20) \quad \sum_{k \in N\mathbb{Z}^d} \left(\sup_{|n|_\infty \geq |k|_\infty} |W_N(n)| \right) < \infty.$$

Given any $i \in \mathbb{Z}^d$, let $n(i)$ be the unique integer in $N\mathbb{Z}^d$ with $i \in n(i) + \{0, \dots, N-1\}^d$. Then

$$(4.21) \quad \alpha_{n(i)} \leq \sum_{|i'-i|_\infty < 3N} w(i') \leq (6N)^{dq} A_q(w) w(i)$$

by (4.1). This together with (4.19) implies that for any $c \in \ell^\infty$,

$$\begin{aligned} |c(i)| &\leq (6N)^d (A_q(w))^{1/q} \alpha_{n(i)}^{-1/q} \|\Psi_{n(i)}^N c\|_{q,w} \\ &\leq (6N)^d (A_q(w))^{1/q} \sum_{n' \in N\mathbb{Z}^d} W_N(n(i) - n') \\ &\quad \times \left(\sum_{j \in \mathbb{Z}^d} h((j - n')/N) |(Ac)(j)| \right) \\ &\leq (6N)^d (A_q(w))^{1/q} \\ &\quad \left\{ \sum_{j \in \mathbb{Z}^d} \left(\sum_{\epsilon \in \{-4, \dots, 4\}^d} W_N(n(i - j) + \epsilon N) \right) |(Ac)(j)| \right\} \\ (4.22) \quad &=: \sum_{j \in \mathbb{Z}^d} g(i - j) |(Ac)(j)|. \end{aligned}$$

Then the sequence $\{g(i)\}_{i \in \mathbb{Z}^d}$ just defined satisfies all requirements in Lemma 4.5 by (4.20) and (4.22). \square

Now we proceed to prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 3.2, it suffices to prove the conclusion for any infinite matrix $A \in \mathcal{B}$.

By Theorem 3.2,

$$(4.23) \quad \|Ac\|_{q',w'} \leq 2^{2d} 3^{d/q'} (A_{q'}(w'))^{1/q'} \|A\|_{\mathcal{B}} \|c\|_{q',w'} \quad \text{for all } c \in \ell_{w'}^{q'}.$$

Let $\{g(i)\}_{i \in \mathbb{Z}^d}$ be the sequence in Lemma 4.5, and set

$$(4.24) \quad A_0 := \sum_{k \in \mathbb{Z}^d} \left(\sup_{|i|_\infty \geq |k|_\infty} g(i) \right) < \infty.$$

Then

$$\begin{aligned} \|c\|_{q',w'} &\leq \left\| \left(\sum_{j \in \mathbb{Z}^d} g(i - j) |(Ac)(j)| \right)_{i \in \mathbb{Z}^d} \right\|_{q',w'} \\ (4.25) \quad &\leq 2^{2d} 3^{d/q'} A_0 (A_{q'}(w'))^{1/q'} \|Ac\|_{q',w'} \quad \text{for all } c \in \ell^\infty \cap \ell_{w'}^{q'}, \end{aligned}$$

where the first inequality follows from (4.3) and the second inequality holds by Theorem 3.2.

Combining (4.23) and (4.25) proves the $\ell_w^{q'}$ -stability for the infinite matrix $A \in \mathcal{B}$. \square

Finally we prove Corollary 1.3.

Proof of Corollary 1.3. The necessity is well known, while the sufficiency follows from Theorem 3.2 and Corollary 1.6, whose proof will be given in the next section. \square

5. INVERSE-CLOSEDNESS

In this section, we prove Theorem 1.4, Corollaries 1.5 and 1.6, and the following Wiener's lemma for the subalgebra $\mathcal{B}_{p,u}$ of $\mathcal{B}(\ell_w^q)$.

Theorem 5.1. *Let $1 \leq p, q < \infty$, w be a discrete A_q -weight, $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$ be a p -submultiplicative weight matrix that satisfies (3.1), (3.2), (3.3) and*

$$(5.1) \quad M := \sup_{i \in \mathbb{Z}^d} u(i, i) < \infty,$$

and $v := (v(i, j))_{i, j \in \mathbb{Z}^d}$ be a companion weight matrix of the p -submultiplicative weight matrix u that satisfies (3.5). If there exist $D \in (0, \infty)$ and $\theta \in (0, 1)$ such that

$$(5.2) \quad \inf_{N \geq 1} (A_N + B_N(p)t) \leq Dt^\theta \quad \text{for all } t \geq 1$$

where

$$(5.3) \quad A_N := \sum_{|k|_\infty \leq N} \sup_{|k|_\infty \leq |i' - j'|_\infty \leq N} v(i', j')$$

and

$$(5.4) \quad B_N(p) := \left\| \left(\sup_{|i' - j'|_\infty \geq |k|_\infty} v(i', j') (u(i', j'))^{-1} \right)_{|k|_\infty \geq N/2} \right\|_{p/(p-1)},$$

then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$.

One may verify that the weight matrices $((1 + |i - j|)^\alpha)_{i, j \in \mathbb{Z}^d}$ with $\alpha > d(1 - 1/p)$, and $(\exp(|i - j|^\delta))_{i, j \in \mathbb{Z}^d}$ with $\delta \in (0, 1)$, and their companion weight matrices satisfy the conditions on weight matrices required in Theorem 5.1 [43]. Hence we have the following corollary of Theorem 5.1.

Corollary 5.2. *Let $1 \leq p, q < \infty$, w be a discrete A_q -weight, and let u be either $((1 + |i - j|_\infty)^\alpha)_{i, j \in \mathbb{Z}^d}$ with $\alpha > d(1 - 1/p)$ or $(\exp(|i - j|_\infty^\delta))_{i, j \in \mathbb{Z}^d}$ with $\delta \in (0, 1)$. Then $\mathcal{B}_{p,u}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell_w^q)$.*

5.1. Proof of Theorem 1.4. Let $A \in \mathcal{B}$ have an inverse $A^{-1} \in \mathcal{B}(\ell_w^q)$. Then $\|c\|_{q,w} \leq \|A^{-1}\|_{\mathcal{B}(\ell_w^q)} \|Ac\|_{q,w}$ for all $c \in \ell_w^q$, where $\|\cdot\|_{\mathcal{B}(\ell_w^q)}$ is the operator norm on $\mathcal{B}(\ell_w^q)$. Therefore A has ℓ_w^q -stability. By Lemma 4.5, there exists a sequence $\{g(i)\}_{i \in \mathbb{Z}^d}$ such that (4.2) and (4.3) hold.

Write $A^{-1} := (b(i, j))_{i,j \in \mathbb{Z}^d}$, set $c_j := (b(i, j))_{i \in \mathbb{Z}^d}$, and for $l_0 \geq 1$ define $c_j^{l_0} := (b_{l_0}(i, j))_{i \in \mathbb{Z}^d}$, $j \in \mathbb{Z}^d$, where $b_{l_0}(i, j) := b(i, j)$ if $|i - j|_\infty \leq l_0$ and 0 otherwise. Then $c_j^{l_0} \in \ell^\infty \cap \ell_w^q$ and

$$(5.5) \quad \lim_{l_0 \rightarrow +\infty} \|c_j^{l_0} - c_j\|_{q,w} = 0.$$

Applying (4.3) to $c_j^{l_0}$ gives

$$(5.6) \quad |b_{l_0}(i, j)| \leq \sum_{i' \in \mathbb{Z}^d} g(i - i') |(Ac_j^{l_0})(i')|, \quad i \in \mathbb{Z}^d.$$

By (4.2), (5.5), and Theorem 3.2,

$$\begin{aligned} & \sum_{i' \in \mathbb{Z}^d} g(i - i') |(Ac_j^{l_0} - c_j)(i')| \\ & \leq w(i)^{-1/q} \left\| \left(\sum_{i'' \in \mathbb{Z}^d} g(i' - i'') |(Ac_j^{l_0} - c_j)(i'')| \right)_{i' \in \mathbb{Z}^d} \right\|_{q,w} \\ & \leq 2^{4d} 3^{2d/q} w(i)^{-1/q} (A_q(w))^{2/q} \|A\|_{\mathcal{B}} \\ & \quad \times \left\| \left(\sup_{|j'|_\infty \geq |k|_\infty} |g(j')| \right)_{k \in \mathbb{Z}^d} \right\|_1 \|c_j^{l_0} - c_j\|_{q,w} \\ (5.7) \quad & \rightarrow 0 \quad \text{as } l_0 \rightarrow +\infty. \end{aligned}$$

Letting $l_0 \rightarrow +\infty$ in (5.6) and applying (5.7) gives

$$(5.8) \quad |b(i, j)| \leq g(i - j) \quad \text{for all } i, j \in \mathbb{Z}^d.$$

Hence the conclusion $A^{-1} \in \mathcal{B}$ follows from (4.2) and (5.8). \square

5.2. Proof of Corollary 1.5. Write $f(\xi) = \sum_{n \in \mathbb{Z}} a(n) e^{-\sqrt{-1}n\xi}$. Then $A := (a(i - j))_{i,j \in \mathbb{Z}}$ belongs to \mathcal{B} and has bounded inverse in $\mathcal{B}(\ell^2)$. Moreover, $A^{-1} = (b(i - j))_{i,j \in \mathbb{Z}}$ for the sequence $b := (b(n))_{n \in \mathbb{Z}}$ determined by $1/f(\xi) = \sum_{n \in \mathbb{Z}} b(n) e^{-\sqrt{-1}n\xi}$. By Theorem 1.4, $A^{-1} \in \mathcal{B}$ which in turn proves the desired conclusion that $1/f \in A^*(\mathbb{T})$. \square

5.3. Proof of Corollary 1.6. The necessity follows from Theorem 3.2. Now the sufficiency: Let $1 \leq q < \infty$, w be a discrete A_q -weight, and let $A \in \mathcal{B}$ have ℓ_w^q -stability. Then A has ℓ^2 -stability by Theorem 1.2, i.e., there exists a positive constant C such that

$$C^{-1} \|c\|_2 \leq \|Ac\| \leq C \|c\|_2 \quad \text{for all } c \in \ell^2.$$

This implies that A^*A has bounded inverse in $\mathcal{B}(\ell^2)$. On the other hand, A^*A belong to \mathcal{B} by Proposition 2.4 and Theorem 3.2. Therefore

$$(5.9) \quad (A^*A)^{-1} \in \mathcal{B}$$

by Theorem 1.4. Now we prove that $B := (A^*A)^{-1}A^*$ is the desired left inverse of the infinite matrix A in \mathcal{B} . The conclusion that $B \in \mathcal{B}$ follows from (5.9), Proposition 2.4 and Theorem 3.2. From the definition of the infinite matrix B , it defines a left inverse in $\mathcal{B}(\ell^2)$, it belongs to $\mathcal{B}(\ell_w^q)$ by Theorem 3.2 and $B \in \mathcal{B}$, and the set $\ell^2 \cap \ell_w^q$ is dense in ℓ_w^q . Therefore the infinite matrix B is a left inverse in $\mathcal{B}(\ell_w^q)$. \square

5.4. Proof of Theorem 5.1. To prove Theorem 5.1, we need a technical lemma. Similar results are established in [41, 43] for infinite matrices in the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, see also Remark 3.4.

Lemma 5.3. *Let $1 \leq p \leq \infty$. If the weight matrix u satisfies (3.1), (3.2), (3.3), (5.1) and (5.2) for some positive constants $D \in (0, \infty)$ and $\theta \in (0, 1)$, then*

$$(5.10) \quad \|A^2\|_{\mathcal{B}_{p,u}} \leq 2^{2+2/p} 5^{(d-1)/p} D \|A\|_{\mathcal{B}_{p,u}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta} \quad \text{for all } A \in \mathcal{B}_{p,u}.$$

Proof. Let $A := (a(i, j))_{i, j \in \mathbb{Z}^d} \in \mathcal{B}_{p,u}$, and let A_N and $B_N(p)$ be as in (5.3) and (5.4) respectively. Recall that $|a(i, j)| \leq \|A\|_{\mathcal{B}(\ell^2)}$ for all $i, j \in \mathbb{Z}^d$. Then for $1 < p < \infty$,

$$\begin{aligned} & \sum_{k' \in \mathbb{Z}^d} |a(i, k')| u(i, k') |a(k', j)| v(k', j) \\ & \leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)} \sum_{|k'-j|_\infty \leq N} |a(i, k')| u(i, k') v(k', j) \right. \\ & \quad \left. + \sum_{|k'-j|_\infty > N} |a(i, k')| u(i, k') |a(k', j)| v(k', j) \right\} \\ & \leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)} \left(\sum_{|k''-j|_\infty \leq N} v(k'', j) \right)^{(p-1)/p} \right. \\ & \quad \times \left(\sum_{|k'-j|_\infty \leq N} (|a(i, k')| u(i, k'))^p v(k', j) \right)^{1/p} \\ & \quad \left. + \left(\sum_{|k''-j|_\infty > N} |a(k'', j)| v(k'', j) \right)^{(p-1)/p} \right. \\ & \quad \left. \times \left(\sum_{|k'-j|_\infty > N} (|a(i, k')| u(i, k'))^p |a(k', j)| v(k', j) \right)^{1/p} \right\}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& \left\{ \sum_{k \in \mathbb{Z}^d} \sup_{|i-j|_\infty \geq |k|_\infty} \left(\sum_{k' \in \mathbb{Z}^d} |a(i, k')| |u(i, k')| |a(k', j)| |v(k', j)| \right)^p \right\}^{1/p} \\
& \leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)} (A_N)^{(p-1)/p} \right. \\
& \quad \times \left(A_N \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'|_\infty \geq |k|_\infty/2} (|a(i', j')| |u(i', j')|)^p \right. \\
& \quad \left. + \|A\|_{\mathcal{B}_{p,u}}^p \sum_{k \in \mathbb{Z}^d} \sup_{|k|_\infty/2 \leq |i'-j'|_\infty \leq N} |v(i', j')| \right)^{1/p} \\
& \quad + \|A\|_{\mathcal{B}_{p,u}}^{(p-1)/p} (B_N(p))^{(p-1)/p} \\
& \quad \times \left(\|A\|_{\mathcal{B}_{p,u}} B_N(p) \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'|_\infty \geq |k|_\infty/2} (|a(i', j')| |u(i, j')|)^p \right. \\
& \quad \left. + \|A\|_{\mathcal{B}_{p,u}}^{p+1} \left(\sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'|_\infty \geq \max(|k|_\infty/2, N)} \left(\frac{|v(i', j')|}{|u(i', j')|} \right)^{p/(p-1)} \right)^{(p-1)/p} \right)^{1/p} \Big\} \\
& \leq 2^{1+2/p} 5^{(d-1)/p} \|A\|_{\mathcal{B}_{p,u}} \inf_{N \geq 1} \left(\|A\|_{\mathcal{B}(\ell^2)} A_N + B_N(p) \|A\|_{\mathcal{B}_{p,u}} \right) \\
& \leq 2^{1+2/p} 5^{(d-1)/p} D \|A\|_{\mathcal{B}_{p,u}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \left\{ \sum_{k \in \mathbb{Z}^d} \sup_{|i-j|_\infty \geq |k|_\infty} \left(\sum_{k' \in \mathbb{Z}^d} |a(i, k')| |v(i, k')| |a(k', j)| |u(k', j)| \right)^p \right\}^{1/p} \\
& \leq 2^{1+2/p} 5^{(d-1)/p} D \|A\|_{\mathcal{B}_{p,u}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\end{aligned}$$

Combining the above two estimates and applying (3.16) with $B = A$, we then get the desire conclusion (5.10) for $1 < p < \infty$.

The conclusion (5.10) for $p = 1$ and for $p = \infty$ can be established similarly. We omit the details here. \square

Having the above technical lemma, we can combine the arguments in [41, 43] and Wiener's lemma for \mathcal{B} to prove Theorem 5.1.

Proof of Theorem 5.1. Let $A \in \mathcal{B}_{p,u}$ and $A^{-1} \in \mathcal{B}(\ell_w^p)$. Then $A^{-1} \in \mathcal{B} \subset \mathcal{B}(\ell^2)$ by Theorems 1.4 and 3.2. This implies that $C_1 I \leq A^* A \leq C_2 I$ for some positive constants C_1 and C_2 , where A^* is the conjugate

transpose of the matrix A and I is the identity matrix. Now set

$$(5.11) \quad B := I - \frac{2}{C_1 + C_2} A^* A.$$

Then

$$(5.12) \quad \|B\|_{\mathcal{B}(\ell^2)} \leq \frac{C_2 - C_1}{C_2 + C_1} := r_0 < 1.$$

On the other hand, $A^* A \in \mathcal{B}_{p,u}$ by Proposition 2.4 and Theorem 3.2, and $I \in \mathcal{B}_{p,u}$ by (5.1). This shows that

$$(5.13) \quad \|B\|_{\mathcal{B}_{p,u}} < \infty.$$

Given any integer $n \geq 1$, write $n = \sum_{l=0}^{l_0} \epsilon_l 2^l$ with $\epsilon_l \in \{0, 1\}$. Applying Theorem 3.2 and Lemma 5.3 iteratively gives

$$\begin{aligned} \|B^n\|_{\mathcal{B}_{p,u}} &\leq (C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} \|B^{n-\epsilon_0}\|_{\mathcal{B}_{p,u}} \\ &\leq C(C\|B\|_{\mathcal{B}_{p,u}})^{\epsilon_0} (\|B\|_{\mathcal{B}(\ell^2)})^{(1-\theta)\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l} \\ &\quad \times (\|B\|_{\mathcal{B}_{p,u}})^{\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l (1+\theta)} \\ &\leq \dots \leq C^{l_0} (C\|B\|_{\mathcal{B}_{p,u}})^{\sum_{l=0}^{l_0} \epsilon_l (1+\theta)^l} (\|B\|_{\mathcal{B}(\ell^2)})^{\sum_{l=0}^{l_0} \epsilon_l (2^l - (1+\theta)^l)} \\ (5.14) \quad &\leq C^{\log_2 n} (Cr_0^{-1}\|B\|_{\mathcal{B}_{p,u}})^{n^{\log_2(1+\theta)}} r_0^n, \end{aligned}$$

where $C = \max(2^{2+2/p} 5^{(d-1)/p} D, 2^{1+2/p} 5^{(d-1)/p} M_p(u))$. This together with (5.12) and (5.13) shows that

$$\begin{aligned} \|A^{-1}\|_{\mathcal{B}_{p,u}} &= \|(A^* A)^{-1} A^*\|_{\mathcal{B}_{p,u}} \\ &= \frac{C_1 + C_2}{2} \left\| A^* + \left(\sum_{n=1}^{\infty} B^n \right) A^* \right\|_{\mathcal{B}_{p,u}} \\ &\leq \frac{C_1 + C_2}{2} \left\{ \|A^*\|_{\mathcal{B}_{p,u}} + C\|A^*\|_{\mathcal{B}_{p,u}} \right. \\ (5.15) \quad &\quad \left. \times \left(\sum_{n=1}^{\infty} C^{\log_2 n} (Cr_0^{-1}\|B\|_{\mathcal{B}_{p,u}})^{n^{\log_2(1+\theta)}} r_0^n \right) \right\} < \infty. \end{aligned}$$

Hence the conclusion $A^{-1} \in \mathcal{B}_{p,u}$ is proved. \square

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